Math 2150

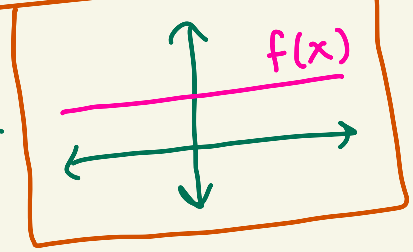
Topic 14 - Laplace Transforms

<u>Def:</u> Given a function f that is defined on $[0, \infty)$, define the Laplace transform of f to be

$$\mathcal{L}[f] = \int_0^\infty e^{-sx} f(x)\,dx$$

Note that $\mathcal{L}[f]$ is a function of s. We will write $\mathcal{L}[f](s)$ when we want to plug s into $\mathcal{L}[f]$.

<u>Ex:</u> Let $f(x) = 1$ for all x.



Then, $\mathcal{L}[f] = \int_0^\infty e^{-sx} \cdot 1 \cdot dx =$

$$= \int_0^\infty e^{-sx}\,dx$$

So, $\mathcal{L}[f](1) = \int_0^\infty e^{-1 \cdot x}\,dx = \int_0^\infty e^{-x}\,dx$

$$= \lim_{T \to \infty} \int_0^T e^{-x}\,dx = \lim_{T \to \infty} \left[-e^{-x} \Big|_0^T \right]$$

$$= \lim_{T \to \infty} \left[-e^{-T} - (-e^{-0}) \right] = 0 + 1 = 1$$

But, $\mathcal{L}[f](-1) = \int_0^\infty e^{-(-1)x}\,dx = \int_0^\infty e^{x}\,dx$

$$= \lim_{T\to\infty} \int_0^T e^x\, d = \lim_{T\to\infty}\left[e^x \Big|_0^T \right]$$

$$= \lim_{t\to\infty}\left[e^T - e^0 \right] = \infty$$

So, $\mathcal{L}[f]$ is undefined at $s=-1$.

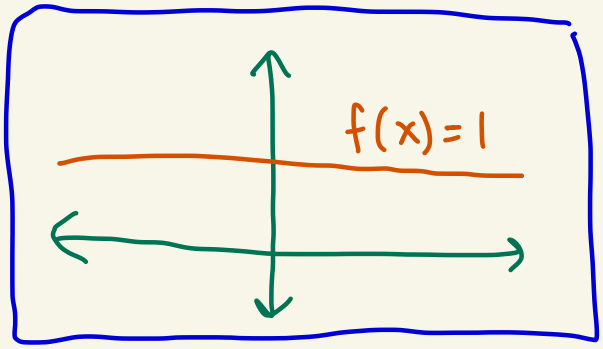
In this case, $\mathcal{L}[f]$ will be defined as long as $s>0$.

If $s>0$, then

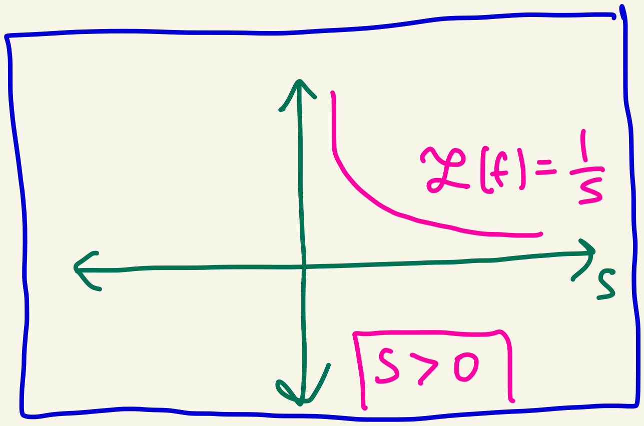
$$\mathcal{L}[f] = \int_0^\infty e^{-sx}\,dx = \lim_{T\to\infty} \int_0^T e^{-sx}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{1}{s} e^{-sx} \Big|_0^T \right] = \lim_{T\to\infty}\left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0} \right]$$

$$= 0 + \frac{1}{s} = \frac{1}{s}.$$



$\xrightarrow{\mathcal{L}}$



Ex: Let $f(x) = e^{ax}$ where a is any constant.
Then if $s > a$ we have

$$\mathcal{L}[f] = \int_0^\infty e^{-sx} f(x)\,dx = \int_0^\infty e^{-sx} e^{ax}\,dx$$

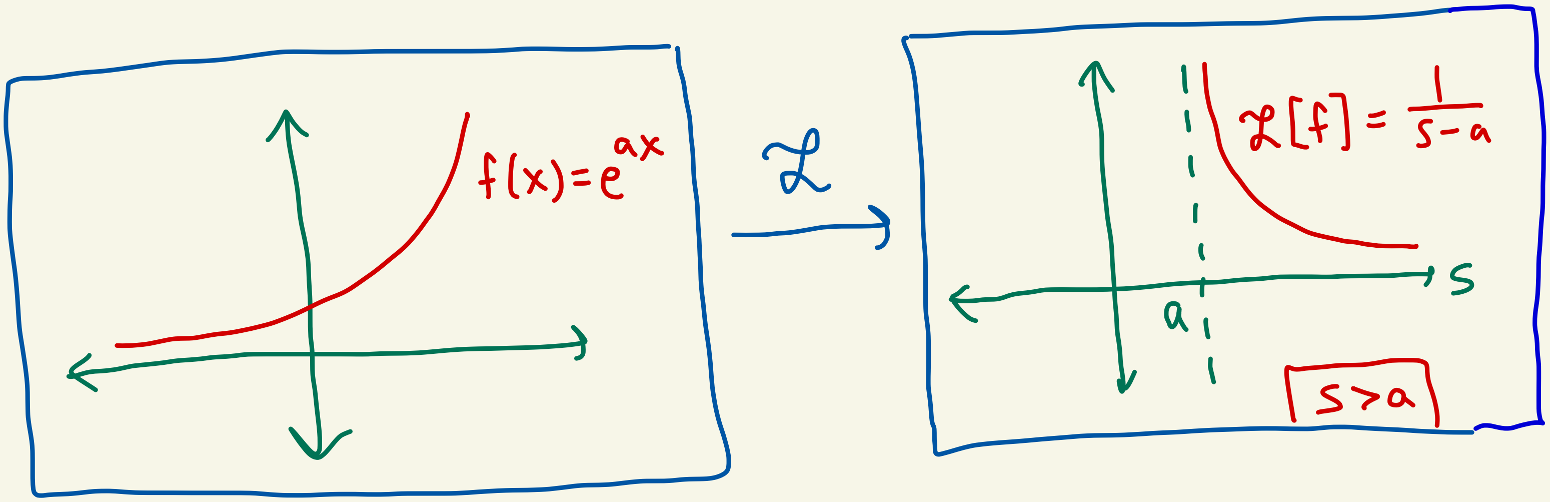
$$= \int_0^\infty e^{(-s+a)x}\,dx = \lim_{T\to\infty} \int_0^T e^{(-s+a)x}\,dx$$

$$= \lim_{T\to\infty} \left[\frac{1}{-s+a} e^{(-s+a)x} \Big|_0^T \right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{-s+a} \underbrace{e^{(-s+a)T}}_{0} - \frac{1}{-s+a} e^{0} \right]$$

since $s > a$
so $-s+a < 0$

$$= 0 - \frac{1}{-s+a} = \frac{1}{s-a}$$



Some Laplace Transforms

① $\mathcal{L}[1] = \frac{1}{s}$ where $s > 0$

② $\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}$ where $s > 0$, $n = 1, 2, 3, 4, \ldots$

③ $\mathcal{L}[e^{ax}] = \frac{1}{s-a}$ where $s > a$

④ $\mathcal{L}[\sin(kx)] = \frac{k}{s^2+k^2}$ where $s > 0$

⑤ $\mathcal{L}[\cos(kx)] = \frac{s}{s^2+k^2}$ where $s > 0$

Theorem:

Suppose $\mathcal{L}[f]$ and $\mathcal{L}[g]$ both exist for $s > s_0$.

Then,

$$\mathcal{L}[c_1 f + c_2 g] = c_1 \mathcal{L}[f] + c_2 \mathcal{L}[g] \quad \text{for } s > s_0$$

Furthermore if f, f', f'' are continuous on $[0, \infty)$ and their Laplace transforms exist then

$$\mathcal{L}[f'] = s \cdot \mathcal{L}[f] - f(0)$$

$$\mathcal{L}[f''] = s^2 \cdot \mathcal{L}[f] - s f(0) - f'(0)$$

Ex: Solve

$$\boxed{\begin{aligned} &y' + 2y = e^{-x} \\ &y(0) = 2 \end{aligned}}$$

using Laplace transforms.

Suppose the laplace transform exists for the solution of the above.

Let $Y(s) = \mathcal{L}[y]$.

Then,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[e^{-x}]$$

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = \frac{1}{s+1}$$

$$\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = \frac{1}{s+1}$$

$$sY(s) - 2 + 2Y(s) = \frac{1}{s+1}$$

$$(s+2)Y(s) = \frac{1}{s+1} + 2$$

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{2}{s+2}$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{2}{s+2}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

$\underline{s=-2}$: $1 = A(0) + B(-1)$

$B = -1$

$\underline{s=-1}$: $1 = A(1) + B(0)$

$1 = A$

$$Y(s) = \frac{1}{s+1} + \frac{1}{s+2}$$

We need some function y whose Laplace transform satisfies

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{s+2}$$

Using the table we have

$$y(x) = e^{-x} + e^{-2x}$$

$$L\left[e^{-x} + e^{-2x}\right] = L\left[e^{-x}\right] + L\left[e^{-2x}\right] = \frac{1}{s+1} + \frac{1}{s+2}$$

So the solution is

$$\boxed{y(x) = e^{-x} + e^{-2x}}$$

Ex: Solve

$$y'' + 4y = 5e^{-x}$$

$$y(0) = 2, \quad y'(0) = 3$$

using Laplace transforms.

Suppose the Laplace transform exists for the solution of the above.

Let $Y(s) = \mathcal{L}[y]$.

Then,

$$\mathcal{L}[y'' + 4y] = \mathcal{L}[5e^{-x}]$$

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = 5\mathcal{L}[e^{-x}]$$

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 4\mathcal{L}[y] = 5 \cdot \frac{1}{s+1}$$

$$s^2Y(s) - 2s - 3 + 4Y(s) = \frac{5}{s+1}$$

$$Y(s)[s^2 + 4] = \frac{5}{s+1} + 2s + 3$$

$$Y(s) = \frac{5}{(s^2+4)(s+1)} + \frac{2s+3}{s^2+4}$$

$$\frac{5}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

$$5 = (As+B)(s+1) + C(s^2+4)$$

$\underline{s=-1:}$ $5 = 0 + C(5) \to C = 1$

$\underline{s=0:}$ $5 = B(1) + (1)(4)$

$1 = B$

$\underline{s=1:}$ $5 = (A+1)(2) + (1)(5)$

$-1 = A$

$$Y(s) = \left(\frac{-s}{s^2+4} + \frac{1}{s^2+4} + \frac{1}{s+1}\right) + \left(\frac{2s}{s^2+4} + \frac{3}{s^2+4}\right)$$

$$Y(s) = 4\frac{1}{s^2+4} + \frac{1}{s+1} + \frac{s}{s^2+4}$$

What $y(x)$ has $\mathcal{L}[y]$ equal to the above?

The answer is:

$$y(x) = 2\sin(2x) + e^{-x} + \cos(2x)$$

Check:

$$\mathcal{L}\left[2\sin(2x) + e^{-x} + \cos(2x)\right]$$

$$= 2\left(\frac{2}{s^2+2^2}\right) + \frac{1}{s-(-1)} + \frac{s}{s^2+2^2}$$

$$= 4\,\frac{1}{s^2+4} + \frac{1}{s+1} + \frac{s}{s^2+4}$$